Returns to origin of a one-dimensional random walk visiting each site an even number of times.

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Abstract

The class of random walks in one dimension, returning to the origin, restricted by the requirement that any site visited (different from the origin) is visited an even number of times, is analyzed in the present note. We call this class the **even-visiting random walks** and provide a closed expression to evaluate them.

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1 Introduction

The number of random walks in one dimension that originate at a given point, we may call the origin, and after $2n$ random steps of unit lenght to the right or to the left, return at the origin (not necessarely for the first time) is $(2n)!/(n!)^2$. However, if we select among them the walks where each site different from the origin is visited an even number of times, the walks have to consist of a number of steps multiple of 4 and their number is more limited. Let us call such random walks **the even-visiting walks**. In these walks the origin is visited an odd number of times, if we count both the end points of the random walk. The purpose of this note is the description of our evaluation of the number of the even-visiting walks and its asymptotics ,which is given in next section. In the rest of this section, we describe how these random walks are related the coefficients of the perturbative expansion of the resolvent of a real non-symmetric random matrix.
We consider the tridiagonal matrix M of order N of the form

$$M = \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & x_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & x_3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & x_4 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & x_5 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & x_{N-1} \\ x_N & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{1.1}$$

The entries x_i , $i = 1, 2, .., N$, are independent random variables with the same probability distribution $P(x) = \frac{1}{2}(\delta(x-1) + \delta(x+1))$. The evaluation of the distribution of complex eigenvalues of the matrix M in the large N limit is a non trivial problem, suggested to us by prof. A.Zee. A numerical exploration of such distribution of eigenvalues , for $N = 400$ was presented in [1], in the context of investigations of one dimensional Schrodinger equation on a chain, where the hopping term may have a random amplitude.
The resolvent $G(z)$ of the random matrix M is usually defined

$$G(z) = \frac{1}{N} \mathrm{Tr} < \frac{1}{z - M} > \tag{1.2}$$

where the expectation value is the mean value with the independent identically distributed random variables x_i :

$$< f(M) >= \int \left[\prod_{i=1}^{N} P(x_i)\, dx_i \right] f(M) \tag{1.3}$$

The formal perturbative expansion of $G(z)$ is

$$G(z) = \frac{1}{N}\sum_{k=0}^{\infty}\frac{< \mathrm{Tr} M^k >}{z^{k+1}} \tag{1.4}$$

If the order N of the matrix is greater than k , as we shall always suppose , since we are interested in the limit $N \to \infty$, any term on the diagonal of $< M^k >_{rr}$ has the same value, then the trace merely cancels the $1/N$ factor. Let us consider the term $k = 8$

$$\sum_{a,b,c,d,e,f,g} < M_{ra}M_{ab}M_{bc}M_{cd}M_{de}M_{ef}M_{fg}M_{gr} > \tag{1.5}$$

By recalling that the non vanishing matrix elements are $M_{ij} = 1$ if $j = i-1$, $M_{ij} = x_i$ if $j = i+1$, each term in the sum (1.5) corresponds to a walk of 8 steps, originating and ending at site r, with 4 steps up and 4 steps down. For instance, the sequence indicated in Fig.1, $M_{r,r+1}M_{r+1,r+2}M_{r+2,r+1}M_{r+1,r+2}M_{r+2,r+1}M_{r+1,r}M_{r,r+1}M_{r+1,r} = x_r \cdot x_{r+1} \cdot 1 \cdot x_{r+1} \cdot 1 \cdot 1 \cdot x_r \cdot 1 = 1$ while the sequence in Fig.2 is $x_r \cdot x_{r+1} \cdot 1 \cdot 1 \cdot x_r \cdot 1 \cdot x_r \cdot 1 = x_r \cdot x_{r+1}$. Each walk corresponding to a product of random variables $\prod_j (x_j)^{n_j}$ where all the powers n_j are even numbers, yields a contribution $+1$, while the walks where one or several random variables occurr with odd exponent are averaged to zero. Then $\frac{1}{N} < \mathrm{Tr} M^k >$ is equal to the number of even visiting walks from a fixed site r to the same site r , of k steps. Because the number of steps up has to be even, the total number of steps k is multiple of 4 and we rewrite eq.(1.4) as

$$G(z) = \sum_{k=0}^{\infty}\frac{c_k}{z^{4k+1}} \quad ; \quad c_k = \lim_{N\to\infty}\frac{1}{N} < \mathrm{Tr}\, M^{4k} > \tag{1.6}$$

2 Evaluation of the number of even visiting walks.

Let $N(2n_r, 2n_{r+1}, \ldots, 2n_{r+j})$ be the number of even visiting walks corresponding to the multiplicity of the product $(x_r)^{2n_r}(x_{r+1})^{2n_{r+1}} \ldots (x_{r+j})^{2n_{r+j}}$. Each walk in this set visits only sites $s \geq r$, the length of the walk is $l = 4(n_r + n_{r+1} + \ldots + n_{r+j})$. The "maximum site" visited is the site $r+j+1$, visited $2n_{r+j}$ times ; the "minimum site" visited is the site r , visited $2n_r + 1$ times. The number $N(2n_r, 2n_{r+1}, \ldots, 2n_{r+j}, 2n_{r+j+1})$ is related to the previous one $N(2n_r, 2n_{r+1}, \ldots, 2n_{r+j})$ as it follows : new walks of length two corresponding to $(x_{r+j+1} \cdot 1)$ may be inserted in each of the maxima of the previous walk. Since $2n_{r+j+1}$ identical objects are placed in $2n_{r+j}$ places in $\binom{2n_{r+j+1}+2n_{r+j}-1}{2n_{r+j+1}}$ ways, we obtain

$$N(2n_r, 2n_{r+1}, \ldots, 2n_{r+j}, 2n_{r+j+1}) =$$

$$= \binom{2n_{r+j+1} + 2n_{r+j} - 1}{2n_{r+j+1}} N(2n_r, 2n_{r+1}, \ldots, 2n_{r+j}) \tag{2.1}$$

By iterating eq.(2.1) with the initial condition $N(2n_r) = 1$ one obtains

$$N(2n_r, 2n_{r+1}, \ldots, 2n_{r+j}) = \prod_{i=1}^{j-1} \binom{2n_{r+i+1} + 2n_{r+i} - 1}{2n_{r+i+1}} \tag{2.2}$$

In the same way the number $N(2n_{r-1}, 2n_r, 2n_{r+1}, \ldots, 2n_{r+j})$ may be evaluated from $N(2n_r, 2n_{r+1}, \ldots, 2n_{r+j})$. Here the walks of the first number are obtained by inserting $2n_{r-1}$ walks of length two $(1 \cdot x_{r-1})$ in each of the $2n_r + 1$ minima of the walks of the second number, obtaining

$$N(2n_{r-1}, 2n_r, 2n_{r+1}, \ldots, 2n_{r+j}) =$$
$$= \binom{2n_{r-1} + 2n_r}{2n_{r-1}} N(2n_r, 2n_{r+1}, \ldots, 2n_{r+j}) \tag{2.3}$$

Each walk contributing to $N(2n_{r-1}, 2n_r, 2n_{r+1}, \ldots, 2n_{r+j})$ visits $2n_{r-1}$ times the "minimum site" $r-1$, then the iteration of the procedure leads to

$$N(2n_{r-s}, 2n_{r-s+1}, \cdots, 2n_{r-1}, 2n_r, 2n_{r+1}, \ldots, 2n_{r+j}) =$$
$$= \left[\prod_{p=0}^{s-2} \binom{2n_{r-s+p} + 2n_{r-s+p+1} - 1}{2n_{r-s+p}}\right] \binom{2n_{r-1} + 2n_r}{2n_{r-1}} \left[\prod_{i=1}^{j-1} \binom{2n_{r+i+1} + 2n_{r+i} - 1}{2n_{r+i+1}}\right] \tag{2.4}$$

The coefficient c_k , we wish to evaluate, is the sum of several multiplicities $N(2n_{r-s}, 2n_{r-s+1}, \cdots, 2n_{r-1}, 2n_r, 2n_{r+1}, \ldots, 2n_{r+j})$ given above , where $k = 2n_{r-s} + 2n_{r-s+1} + \cdots + 2n_{r+j}$ corresponds to even visiting walks of $4k$ steps.

The evaluation may be someway simplified, by considering walks of fixed width, that is the difference between the "maximum site" visited and the "minimum site" visited . We consider the set of ordered partitions of k into positive integers $[n_1, n_2, \cdots, n_t]$ where $k = \sum n_p$. To each ordered sequence , or more properly to each composition , $[n_1, n_2, \cdots, n_t]$ corresponds $t+1$ classes of even visiting walks, which are associated to the products

$$\begin{array}{l} (x_{r-t})^{2n_1} (x_{r-t+1})^{2n_2} \cdots (x_{r-1})^{2n_t} \quad ; \\ (x_{r-t+1})^{2n_1} (x_{r-t+2})^{2n_2} \cdots (x_r)^{2n_t} \quad ; \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad ; \\ (x_r)^{2n_1} (x_{r+1})^{2n_2} \cdots (x_{r+t-1})^{2n_t} \end{array} \tag{2.5}$$

All walks in eq.(2.5) have the same width $w = t$. Their multiplicities, given in eq.(2.4) are simply related and their sum is

$$S_{[n_1, n_2, \cdots, n_t]} = \frac{2k}{n_1} \prod_{i=1}^{j-1} \binom{2n_{r+i+1} + 2n_{r+i} - 1}{2n_{r+i+1}} \tag{2.6}$$

Next we sum over the ordered partitions that correspond to the permutations of the positive integers $[n_1, n_2, \cdots, n_t]$, finally over the different widths, from 1 to k, that is c_k is the sum over the 2^{k-1} compositions :

$$c_k = \sum_{t=1}^{k} \sum_{\text{perm.}} S_{[n_1,n_2,\cdots,n_t]} = \sum_{\text{comp.}} S_{[n_1,n_2,\cdots,n_t]} \tag{2.7}$$

The evaluation of eq.(2.7) may be automated and we find the first coefficients c_k :

c_0	1
c_1	2
c_2	14
c_3	116
c_4	1 110
c_5	11 372
c_6	123 020
c_7	1 384 168
c_8	16 058 982
c_9	190 948 796
c_{10}	2 317 085 924
c_{11}	28 602 719 576
c_{12}	358 298 116 092
c_{13}	4 545 807 497 272
c_{14}	58 321 701 832 408
c_{15}	755 700 271 652 816
c_{16}	9 878 971 460 641 414

(2.8)

The ratios c_n/c_{n-1} rise monotonically with a rate slower at higher values of n. Dr.L.Molinari provided us a proof [2] that the eigenvalues of the random matrix M , (1.1), are inside the square with vertices $(\pm 2, 0)$, $(0, \pm 2i)$. This implies that $c_n \sim 16^n$. Let us represent the asymptotic behaviour of r_N , the number of returns to the origin of unrestricted random walks of N steps, $r_N \sim A\,\mu^N\,N^{\gamma-1}$, then $\mu = 2$, $\gamma = 1/2$. The previous asymptotic behaviour for c_n implies that $\mu = 2$ also for the returns to the origin of even-visiting random walks.

References

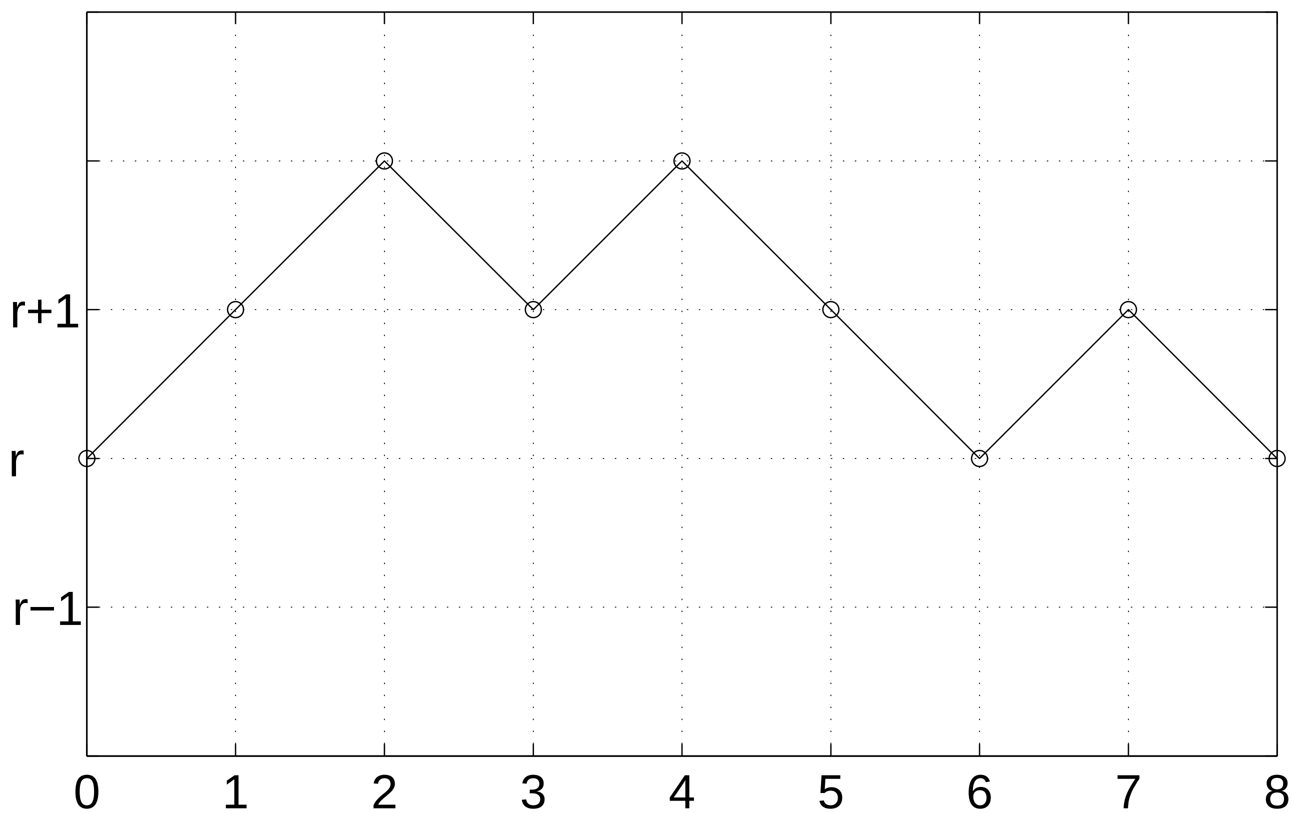
[1] J.Feinberg, A.Zee, Non-hermitian localization and de-localization, cond-mat/9706218.

[2] We plan to provide this proof together with the analysis of the distribution of the eigenvalues in the near future.

Figure Captions

Fig.1 One of the even-visiting random walks, returning to site r after 8 steps, with width $w = 2$.

Fig.2 One random walk not belonging to the class of even-visiting random walks.



r+1
r
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2
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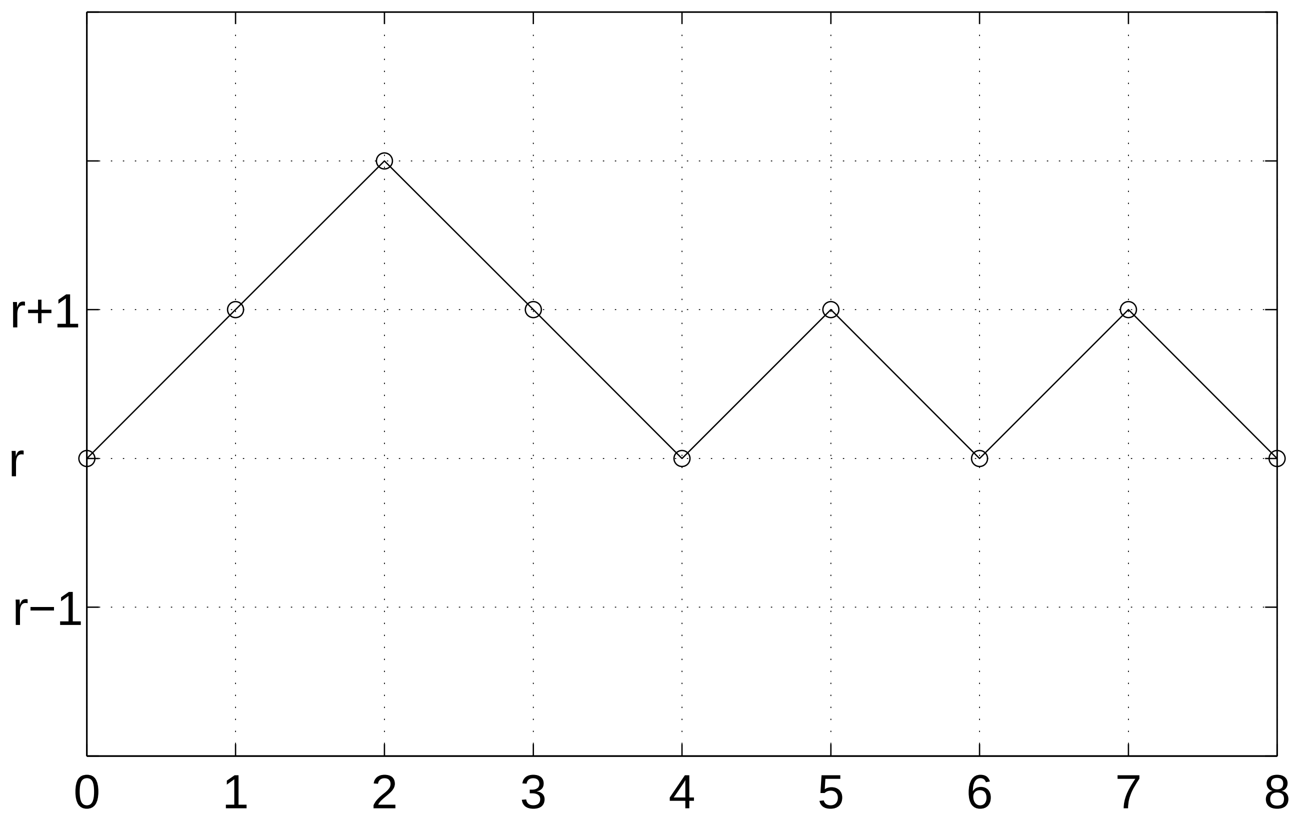
r+1
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